Complex Numbers

Imaginary Number: $i=\sqrt{-1}$ and $i^2=-1$

Complex Numbers: The set of all numbers of the form $a+bi$ where a and b are real numbers.

a is called the **real part** and b is called the **imaginary part**. If $b \neq 0$, then $a+bi$ is an **imaginary number.** The form $a+bi$ is called the **standard form** of a complex number.

Examples: Determine whether each complex number is real or imaginary and write it in standard form.

a) $4i$
imaginary
$0+4i$

b) $3-6i$
imaginary
$3-6i$

c) 5
real
$5+0i$

d) $\frac{i-3\pi}{4}$
imaginary
$-\frac{3\pi}{4}+\frac{1}{4}i$

Addition, Subtraction, and Multiplication of Complex Numbers.

$(a+bi)+(c+di)=(a+c)+(b+d)i$

$(a+bi)-(c+di)=(a-c)+(b-d)i$

$(a+bi)(c+di)=(ac-bd)+(bc+ad)i$ or use FOIL.

Examples: Perform the indicated operations.

a) $(6+2i)+(4-3i)=6+4+2i-3i$
$=10-i$

b) $(7-4i)-(-2+8i)=7+2-4i-8i$
$=9-12i$

c) $(6+5i)(8+3i)=$
$48+18i+40i+15i^2=$
$48+58i+15(-1)=$
$\boxed{33+58i}$

d) $(1-i)(4+i)=4+i-4i-i^2$
$=4-3i-(-1)$
$=\boxed{5-3i}$

Powers of *i*:

Since $i^2=-1$, $i^3=i^2\cdot i=-1\cdot i=-i$, and $i^4=i^2\cdot i^2=-1\cdot -1=1$

The first eight powers are listed here:

$i^1=i$ $\quad i^5=i$

$i^2=-1$ $\quad i^6=-1$

$i^3=-i$ $\quad i^7=-i$

$i^4=1$ $\quad i^8=1$

The powers of i continue in this pattern.

Exponent $\div 4$

Remainder	Simplifies To:
0	1
1	i
2	-1
3	$-i$

Examples: Simplify the power of i.

a) $i^{35} = i^{34} \cdot i$
$= (i^2)^{17} \cdot i$
$= (-1)^{17} \cdot i$
$= \boxed{-i}$

b) $i^{29} = i^{28} \cdot i$
$= (i^2)^{14} \cdot i$
$= (-1)^{14} \cdot i$
$= \boxed{i}$

c) $i^{98} = (i^2)^{49}$
$= (-1)^{49}$
$= \boxed{-1}$

d) $i^{48} = (i^2)^{24}$
$= (-1)^{24}$
$= \boxed{1}$

Theorem: If a and b are real numbers, then the product of $a+bi$ and its conjugate $a-bi$ is the real number a^2+b^2. $(a+bi)(a-bi) = (a^2+b^2)$

Examples: Find the product of the complex number and its conjugate.

a) $3-7i$
$(3-7i)(3+7i)$
$= 3^2+7^2 = \boxed{58}$

b) $2+9i$
$(2+9i)(2-9i)$
$= 2^2+9^2 = \boxed{85}$

c) i
$i \cdot (-i)$
$= -i^2 = \boxed{1}$

Examples: Write each quotient in the form $a+bi$.

a) $\dfrac{6-2i}{3} =$
$\boxed{2 - \frac{2}{3}i}$

b) $\dfrac{2}{8+9i} =$
$\left(\dfrac{2}{8+9i}\right)\left(\dfrac{8-9i}{8-9i}\right) =$
$\dfrac{16-18i}{8^2+9^2} = \dfrac{16-18i}{145} =$
$\boxed{\dfrac{16}{145} - \dfrac{18}{145}i}$

c) $\dfrac{4-5i}{3+2i} = \left(\dfrac{4-5i}{3+2i}\right)\left(\dfrac{3-2i}{3-2i}\right) =$
$\dfrac{12-8i-15i+10i^2}{3^2+2^2} =$
$\dfrac{12-23i-10}{13} =$
$\dfrac{2-23i}{13} = \boxed{\dfrac{2}{13} - \dfrac{23}{13}i}$

Roots of Negative Numbers

For any positive real number b, $\sqrt{-b} = i\sqrt{b}$.

Examples: Write each expression in the form $a+bi$, where a and b are real numbers.

a) $\sqrt{-5}+\sqrt{-8} =$
$i\sqrt{5} + i\sqrt{8} =$
$i\sqrt{5} + 2i\sqrt{2} =$
$\boxed{(\sqrt{5}+2\sqrt{2})i}$

b) $\sqrt{-20}\left(\sqrt{-6}-\sqrt{-4}\right) =$
$i\sqrt{20}\,(i\sqrt{6}-2i) =$
$2i\sqrt{5}\,(i\sqrt{6}-2i) =$
$2i^2\sqrt{30} - 4i^2\sqrt{5} =$
$\boxed{-2\sqrt{30}+4\sqrt{5}}$

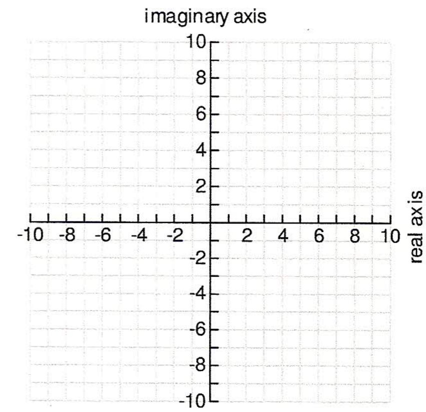
c) $\dfrac{-2+\sqrt{-48}}{2} = \dfrac{-2+i\sqrt{48}}{2}$
$= \dfrac{-2+4i\sqrt{3}}{2} = \boxed{-1+2\sqrt{3}\,i}$

Example: Does the complex number $x = 1+3i\sqrt{2}$ satisfies the equation $x^2-2x+4=0$?

$(1+3i\sqrt{2})^2 - 2(1+3i\sqrt{2}) + 4 \stackrel{?}{=} 0$
$1 + \cancel{6i\sqrt{2}} + (3i\sqrt{2})^2 - 2 - \cancel{6i\sqrt{2}} + 4 \stackrel{?}{=} 0$
$1 + 9 \cdot i^2 \cdot 2 - 2 + 4 \stackrel{?}{=} 0$
$1 - 18 - 2 + 4 \stackrel{?}{=} 0$
$-15 \neq 0$

$\boxed{\text{not a solution}}$

Trigonometric Form of Complex Numbers

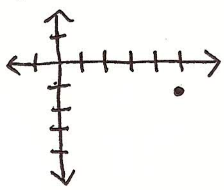


The complex number $a+bi$ can be thought of as an ordered pair (a,b). We graph it on the **complex plane** where the horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis.**

Absolute Value or Modulus: $|a+bi|=\sqrt{a^2+b^2}$. (The distance between the number and the origin on the complex plane.)

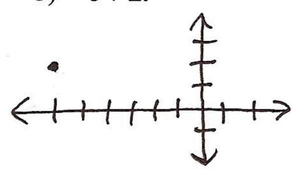
Examples: Graph each complex number and find its absolute value.

a) $5-i$



$|5-i|=\sqrt{5^2+(-1)^2}$
$=\boxed{\sqrt{26}}$

b) $-6+2i$



$|-6+2i|=\sqrt{(-6)^2+2^2}$
$=\sqrt{40}=\boxed{2\sqrt{10}}$

Trigonometric Form of a Complex Number
If $z=a+bi$ is a complex number, then the trigonometric form of z is

$$z=r(\cos\theta+i\sin\theta),\ \text{sometimes abbreviated } z=r\operatorname{cis}\theta,$$

where r is called the **modulus** and θ is called the **argument,** defined as the angle in standard position whose terminal side contains the point (a,b).

$$r=\sqrt{a^2+b^2}$$
$$a=r\cos\theta \text{ and } b=r\sin\theta.$$

We usually use the smallest possible nonnegative angle for θ.

Examples: Write each complex number in trigonometric form.

a) $-2\sqrt{3}+2i$ QII

$r=\sqrt{(-2\sqrt{3})^2+2^2}=4$

$\cos\theta=\frac{-2\sqrt{3}}{4}=-\frac{\sqrt{3}}{2}$

$\theta=\frac{5\pi}{6}$

$\boxed{4\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right) \text{ or } 4\operatorname{cis}\left(\frac{5\pi}{6}\right)}$

b) $5-4i$ QIV

$r=\sqrt{5^2+(-4)^2}=\sqrt{41}$

$\cos\theta=\frac{5}{\sqrt{41}}$ $\sin\theta=\frac{-4}{\sqrt{41}}$

$\theta\approx 321.3^\circ$

$\boxed{\sqrt{41}(\cos 321.3^\circ+i\sin 321.3^\circ) \text{ or } \sqrt{41}\operatorname{cis}321.3^\circ}$

Example: Write the complex number $12\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ in the form $a+bi$.

$$12\left(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)=\boxed{6\sqrt{3}+6i}$$

Product and Quotient of Complex Numbers

If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$, and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then

$$z_1z_2=r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

Examples: Find the product and quotient using trigonometric form.

$$z_1=4\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right),\quad z_2=8\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

a) Find z_1z_2

$$z_1z_2=4\cdot 8\left(\cos(\pi/12+\pi/12)+i\sin(\pi/12+\pi/12)\right)$$

$$=\boxed{32\left(\cos(\pi/6)+i\sin(\pi/6)\right)}$$

$$=32\left(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)=\boxed{16\sqrt{3}+16i}$$

b) Find $\frac{z_1}{z_2}$

$$\frac{z_1}{z_2}=\frac{4}{8}\left(\cos(\pi/12-\pi/12)+i\sin(\pi/12-\pi/12)\right)$$

$$=\boxed{\frac{1}{2}(\cos 0+i\sin 0)}$$

$$=\frac{1}{2}(1+0i)$$

$$=\boxed{\frac{1}{2}}$$

Complex Conjugates

The conjugate of $r(\cos(\theta)+i\sin(\theta))$ is $r(\cos(-\theta)+i\sin(-\theta))$

Example: Find the product of the following and its conjugate: $6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

Conjugate: $6\left(\cos(-\pi/3)+i\sin(-\pi/3)\right)$

$$6\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\cdot 6\left(\cos(-\pi/3)+i\sin(-\pi/3)\right)=$$

$$6\cdot 6\left(\cos(\pi/3+(-\pi/3))+i\sin(\pi/3+(-\pi/3))\right)=$$

$$36(\cos 0+i\sin 0)=$$

$$36(1+0i)=\boxed{36}$$

Powers and Roots of Complex Numbers

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is any positive integer, then

$$z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Examples:

a) Simplify $(1+i)^6$.

$1+i \quad r = \sqrt{1^2+1^2} = \sqrt{2}$

QI $\quad \cos\theta = \frac{1}{\sqrt{2}} \quad \sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \pi/4$

$\left[\sqrt{2}\left(\cos \pi/4 + i\sin \pi/4\right)\right]^6 =$

$(\sqrt{2})^6\left(\cos(6\cdot\pi/4) + i\sin(6\cdot\pi/4)\right) =$

$8\left(\cos 3\pi/2 + i\sin 3\pi/2\right) =$

$8(0 - i) = \boxed{-8i}$

b) Simplify $\left(\sqrt{3}-i\right)^4$. $\quad r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

QIV $\quad \cos\theta = \frac{\sqrt{3}}{2} \quad \sin\theta = -\frac{1}{2} \quad \theta = \frac{11\pi}{6}$

$\left[2\left(\cos 11\pi/6 + i\sin 11\pi/6\right)\right]^4 =$

$2^4\left(\cos(4\cdot 11\pi/6) + i\sin(4\cdot 11\pi/6)\right) =$

$16\left(\cos 22\pi/3 + i\sin 22\pi/3\right) =$

$16\left(\cos 4\pi/3 + i\sin 4\pi/3\right) =$

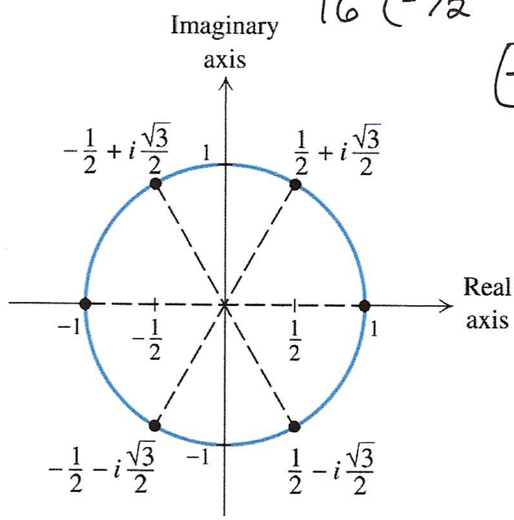
$16\left(-1/2 - \sqrt{3}/2\, i\right) =$

$\boxed{-8 - 8\sqrt{3}i}$

Roots of a Complex Number

How many square roots does 4 have?
How many square roots does -9 have?
How many sixth roots does 1 have?

It turns out that 1 has 6 sixth roots, and they are distributed evenly around the complex plane.



Sixth Roots of Unity

The complex number $a+bi$ is an nth root of the complex number z if $(a+bi)^n = z$.

For any positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots given by the expression $r^{1/n}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$ for $k = 0, 1, 2, \ldots, n-1$.

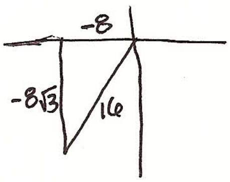
In radians, the roots are given by $r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$ for $k = 0, 1, 2, \ldots, n-1$.

The first of the n roots has an argument of $\frac{\theta}{n}$, and the other roots are spaced $\left(\frac{360}{n}\right)^\circ$ apart.

(The circle is divided evenly into n pieces.)

Examples:

a) Find all of the fourth roots of the complex number $-8-8i\sqrt{3}$.



$r=\sqrt{(-8)^2+(-8\sqrt{3})^2}=16$

$\cos\theta=-8/16=-1/2 \quad \theta=4\pi/3$

$n=4$

$\dfrac{\theta+2k\pi}{n}=\dfrac{4\pi/3+2k\pi}{4}$

$=\pi/3+k\pi/2$

$\pi/3,\ 5\pi/6,\ 4\pi/3,\ 11\pi/6$

$16^{1/4}(\cos\pi/3+i\sin\pi/3)$
$=2(1/2+\sqrt{3}/2\,i)$
$=\boxed{1+i\sqrt{3}}$

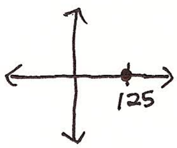
$16^{1/4}(\cos 5\pi/6+i\sin 5\pi/6)$
$=2(-\sqrt{3}/2+1/2\,i)$
$=\boxed{-\sqrt{3}+i}$

$16^{1/4}(\cos 4\pi/3+i\sin 4\pi/3)$
$=2(-1/2-\sqrt{3}/2\,i)$
$=\boxed{-1-i\sqrt{3}}$

$16^{1/4}(\cos 11\pi/6+i\sin 11\pi/6)$
$=2(\sqrt{3}/2-1/2\,i)$
$=\boxed{\sqrt{3}-i}$

b) Find all the cube roots of 125.

$125+0i$



$r=125$
$\theta=0°$

$125(\cos 0°+i\sin 0°)$

$n=3$

$\dfrac{\theta+360°k}{3}=\dfrac{0°+360°k}{3}$

$=120°k$

$=0°,\ 120°,\ 240°$

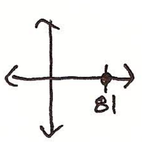
$125^{1/3}(\cos 0°+i\sin 0°)$
$=5(1+0i)=\boxed{5}$

$125^{1/3}(\cos 120°+i\sin 120°)$
$=5(-1/2+\sqrt{3}/2\,i)$
$=\boxed{-5/2+\frac{5\sqrt{3}}{2}i}$

$125^{1/3}(\cos 240°+i\sin 240°)$
$=5(-1/2-\sqrt{3}/2\,i)$
$=\boxed{-5/2-\frac{5\sqrt{3}}{2}i}$

c) Find all complex solutions to $x^4-81=0$.

$x^4=81$ ← 4th roots of 81



$r=81 \quad \theta=0°$

$81(\cos 0°+i\sin 0°)$

$n=4$

$\dfrac{0°+360°k}{4}=\dfrac{360°k}{4}=90°k$

$0°,\ 90°,\ 180°,\ 270°$

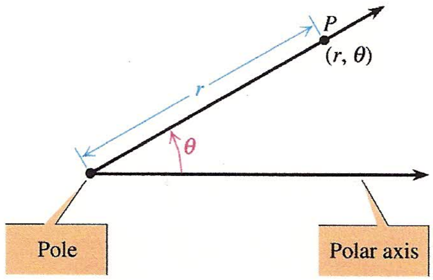
$81^{1/4}(\cos 0°+i\sin 0°)$
$3(1+0i)=\boxed{3}$

$81^{1/4}(\cos 90°+i\sin 90°)$
$3(0+i)=\boxed{3i}$

$81^{1/4}(\cos 180°+i\sin 180°)$
$3(-1+0i)=\boxed{-3}$

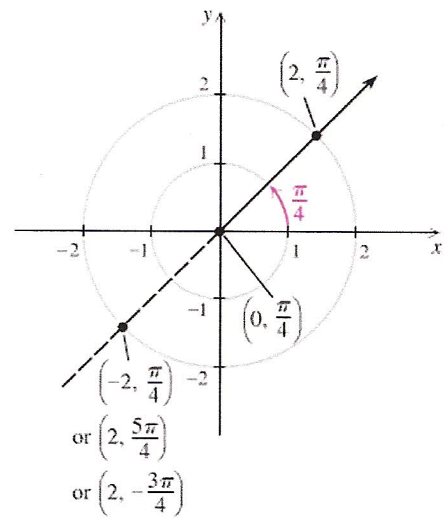
$81^{1/4}(\cos 270°+i\sin 270°)$
$3(0-i)=\boxed{-3i}$

Polar Equations



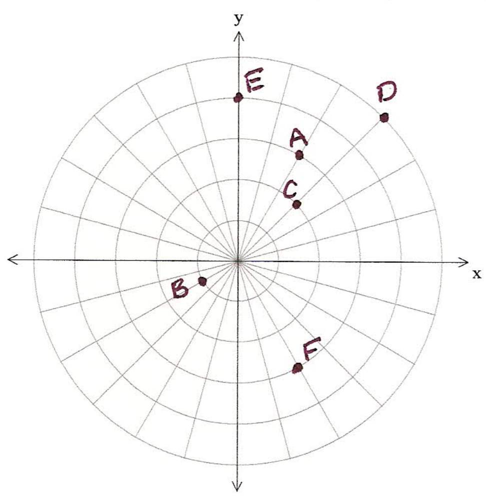
The polar coordinate system is based on a fixed point called the **pole** and a fixed axis called the **polar axis.** Points are represented by ordered pairs in the form (r,θ), where r is the **directed distance** from the pole and θ is an angle whose initial side is the polar axis and whose terminal side contains the point. Typically, we choose the origin as the pole and the positive x-axis as the polar axis.

*To graph $(-r,\theta)$, you move in the opposite direction you would move to graph (r,θ).



Polar coordinates are not unique. The points $\left(-2,\frac{\pi}{4}\right)$, $\left(2,\frac{5\pi}{4}\right)$, and $\left(2,-\frac{3\pi}{4}\right)$ all name the same point.

Examples: Plot the points whose polar coordinates are given.
$A\left(3,\frac{\pi}{3}\right)$, $B\left(-1,\frac{\pi}{6}\right)$, $C\left(2,-\frac{7\pi}{4}\right)$, $D\left(-5,-\frac{3\pi}{4}\right)$, $E\left(4,\frac{\pi}{2}\right)$, $F\left(-3,\frac{2\pi}{3}\right)$



Polar-Rectangular Conversion Rules

- To convert (r,θ) to rectangular coordinates (x,y), use $x = r\cos\theta$ and $y = r\sin\theta$.
- To convert (x,y) to polar coordinates (r,θ), use $r = \sqrt{x^2+y^2}$ and any angle θ in standard position whose terminal side contains (x,y).

Examples:

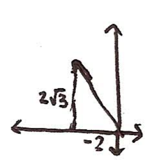
a) Convert $(3,45°)$ to rectangular coordinates.

$x = 3\cos 45° = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$

$y = 3\sin 45° = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$

$\boxed{\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)}$

b) Convert $\left(-2, 2\sqrt{3}\right)$ to polar coordinates.



$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$

$= \sqrt{4+12} = \sqrt{16} = 4$

$\cos\theta = -\frac{2}{4} = -\frac{1}{2}$ $\quad \theta = \frac{2\pi}{3}$

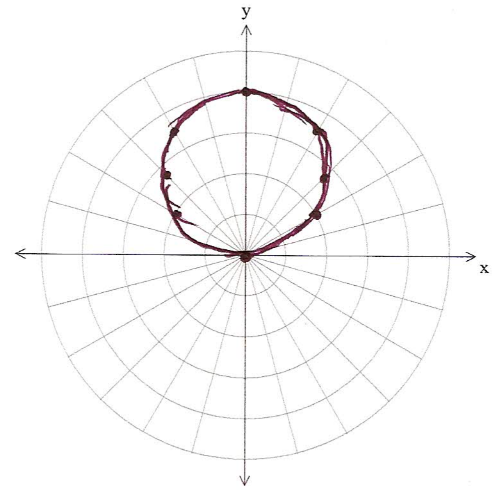
$\boxed{\left(4, \frac{2\pi}{3}\right)}$

Graphing Polar Equations

Examples: Sketch the graphs of the following:

a) $r = 4\sin\theta$

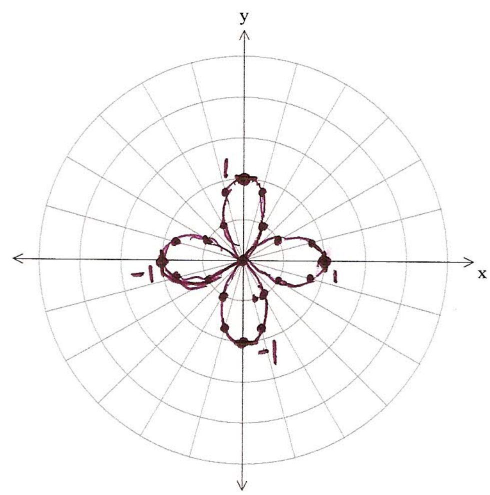
θ	r
0	$4(0)=0$
$\pi/6$	$4(1/2)=2$
$\pi/4$	$4(\sqrt{2}/2)=2\sqrt{2}\approx 2.8$
$\pi/3$	$4(\sqrt{3}/2)=2\sqrt{3}\approx 3.5$
$\pi/2$	$4(1)=4$
$2\pi/3$	$4(\sqrt{3}/2)=2\sqrt{3}\approx 3.5$
$5\pi/6$	$4(1/2)=2$
π	$4(0)=0$



b) $r = \cos(2\theta)$

θ	r
0	1
$\pi/12$	$\sqrt{3}/2 \approx .9$
$\pi/6$	$1/2$
$\pi/4$	0
$\pi/3$	$-1/2$
$5\pi/12$	$-\sqrt{3}/2 \approx -.9$
$\pi/2$	-1

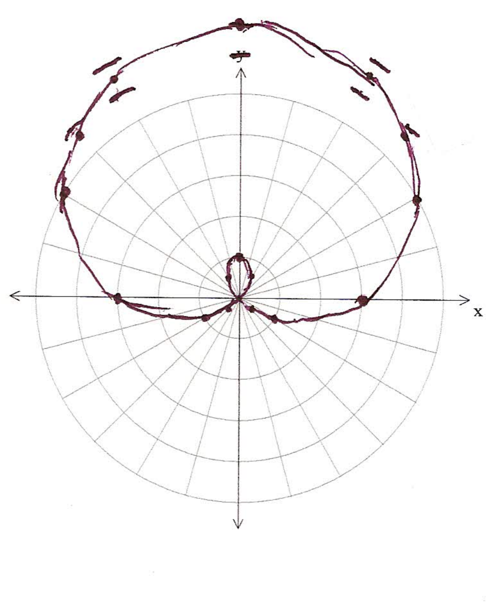
θ	r
$3\pi/4$	0
π	1
$5\pi/4$	0
$3\pi/2$	-1
$7\pi/4$	0
2π	1



c) $r = 3 + 4\sin\theta$

θ	r
0	3
$\pi/6$	5
$\pi/4$	5.8
$\pi/3$	6.5
$\pi/2$	7
$2\pi/3$	6.5
$3\pi/4$	5.8
$5\pi/6$	5
π	3

θ	r
$7\pi/6$	1
$5\pi/4$	$.2$
$4\pi/3$	$-.5$
$3\pi/2$	-1
$5\pi/3$	$-.5$
$7\pi/4$	$.2$
$11\pi/6$	1
2π	3



Use $x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$

Examples: Convert equations from polar to rectangular form.

a) Convert $r = 3\sin\theta$ to a rectangular equation.

$r(r) = r(3\sin\theta)$

$r^2 = 3r\sin\theta$

$\boxed{x^2+y^2=3y}$ ← circle

b) Convert $r = \dfrac{4}{1+\sin\theta}$ to a rectangular equation.

$r(1+\sin\theta)=4$

$r + r\sin\theta = 4$

$r + y = 4$

$r = -y+4$

$r^2 = (-y+4)^2$

$r^2 = y^2-8y+16$

$x^2+y^2 = y^2-8y+16$

$\boxed{x^2 = -8y+16}$ ← parabola

Examples: Convert equations from rectangular to polar form.

c) Convert $y = -2x+5$ to a polar equation.

$r\sin\theta = -2r\cos\theta + 5$

$r\sin\theta + 2r\cos\theta = 5$

$r(\sin\theta + 2\cos\theta) = 5$

$\boxed{r = \dfrac{5}{\sin\theta + 2\cos\theta}}$

d) Convert $x^2+(y-1)^2=1$ to a polar equation.

$x^2+y^2-2y+1=1$

$x^2+y^2-2y=0$

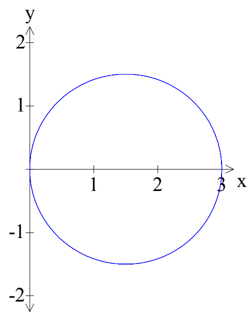
$x^2+y^2=2y$

$r^2 = 2r\sin\theta$

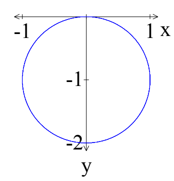
$\boxed{r = 2\sin\theta}$

1) Lines through the origin are of the form $\theta = \alpha$.
Vertical lines are of the form $r\cos\theta = a$.
Horizontal lines are of the form $r\sin\theta = a$.

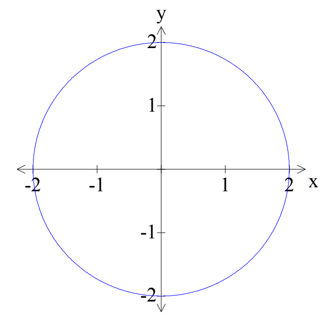
2) Circles come in three forms: $r = a\cos\theta$, $r = a\sin\theta$, and $r = a$.



$r = 3\cos\theta$

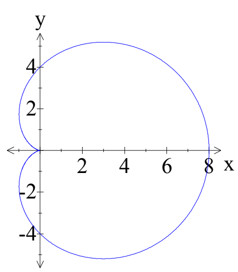


$r = -2\sin\theta$

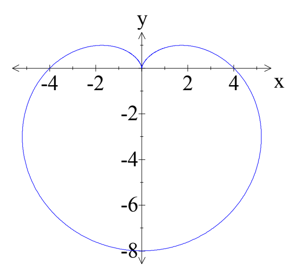


$r = 2$

3) Cardioids have the form $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$.
Cardioids pass through the pole.

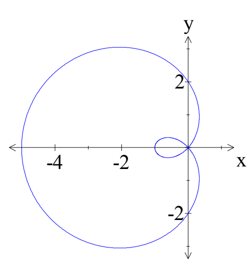


$r = 4 + 4\cos\theta$

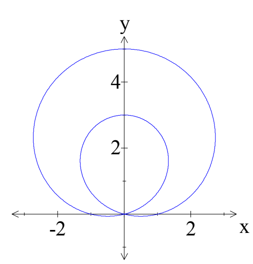


$r = 4 - 4\sin\theta$

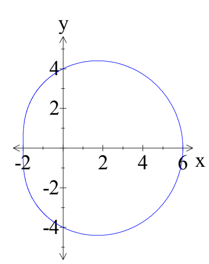
4) Limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$.
Limaçons have an inner loop if $0 < a < b$ and have no inner loop if $0 < b < a$.
The graph of a limaçon with an inner loop passes through the pole twice.
The graph of a limaçon with no inner loop does not pass through the pole.



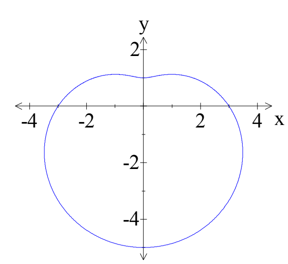
$r = 2 - 3\cos\theta$



$r = 1 + 4\sin\theta$

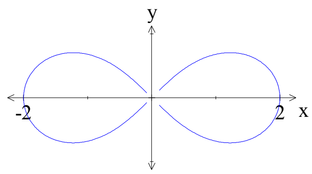


$r = 4 + 2\cos\theta$

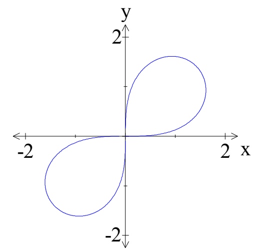


$r = 3 - 2\sin\theta$

5) Lemniscates have the form $r^2 = a^2\cos(2\theta)$ or $r^2 = a^2\sin(2\theta)$.

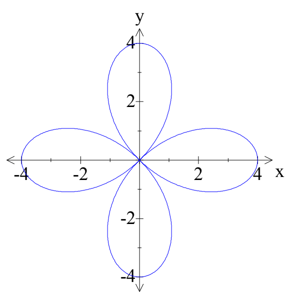


$r^2 = 4\cos(2\theta)$

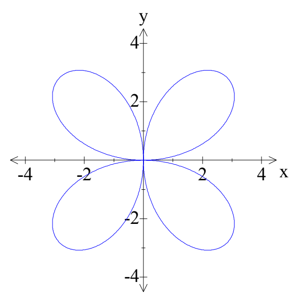


$r^2 = 4\sin(2\theta)$

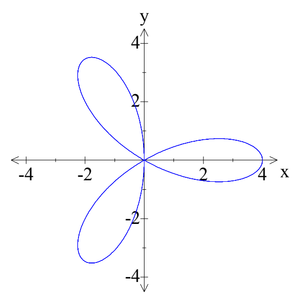
6) Roses have the form $r = a\cos(n\theta) + b$ and $r = a\sin(n\theta) + b$.
If n is even, there are $2n$ loops in the rose.
If n is odd, there are n loops in the rose.



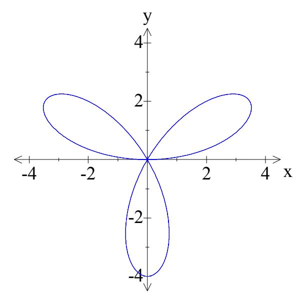
$r = 4\cos(2\theta)$



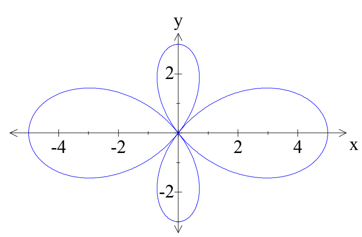
$r = 4\sin(2\theta)$



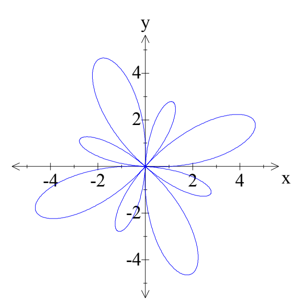
$r = 4\cos(3\theta)$



$r = 4\sin(3\theta)$



$r = 4\cos(2\theta) + 1$



$r = 4\sin(4\theta) + 1$

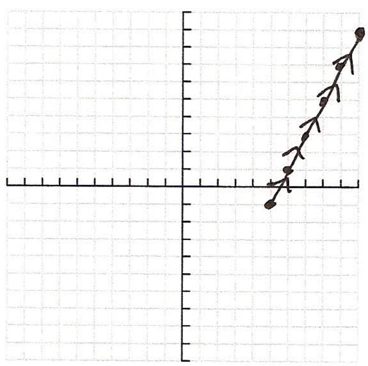
Parametric Equations

Sometimes, it is convenient to express both x and y as functions of a third variable, t. If $f(t)$ and $g(t)$ are both functions of t, where t is some interval of real numbers, then the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations.** The variable t is called the **parameter.** If we think of t as time, then we know when each point of the graph is plotted.

Graphing Parametric Equations

1. Make a t, x, y table for the two equations.
2. Plot the ordered pairs of values of x and y.
3. Mark the **orientation** of the curve by using arrows to show the direction of the graph.

Example: Graph the parametric equations $x = t + 5$ and $y = 2t - 1$ for t in $[0,5]$.



t	x	y
0	5	-1
1	6	1
2	7	3
3	8	5
4	9	7
5	10	9

Graph goes through (5,-1), (6,1), (7,3), (8,5), (9,7), & (10,9)

Eliminating the Parameter

1. Set one equation equal to t.
2. Substitute that equation in for t in the other equation.
3. Sometimes it is more convenient to use a trigonometric identity to eliminate the parameter.

Examples: Eliminate the parameter and identify the graph of the parametric equation.

a) $x = 4t - 9,\ y = -t + 1,\ \ -\infty < t < \infty$

$x + 9 = 4t$

$t = \frac{x+9}{4}$

$t = \frac{1}{4}x + \frac{9}{4}$

$y = -t + 1$

$y = -\left(\frac{1}{4}x + \frac{9}{4}\right) + 1$

$y = -\frac{1}{4}x - \frac{9}{4} + 1$

$\boxed{y = -\frac{1}{4}x - \frac{5}{4}}$

line

b) $x = 5\sin t,\ y = 5\cos t,\ \ -\infty < t < \infty$

$x^2 + y^2 = (5\sin t)^2 + (5\cos t)^2$

$x^2 + y^2 = 25\sin^2 t + 25\cos^2 t$

$x^2 + y^2 = 25(\sin^2 t + \cos^2 t)$

$x^2 + y^2 = 25(1)$

$\boxed{x^2 + y^2 = 25}$

circle

Writing Parametric Equations for Line Segments

1. Write both parametric equations as linear functions: $x = m_1 t + b_1$, and $y = m_2 t + b_2$.
2. Substitute x and t values into the x equation to create a system of equations you can solve for m_1 and b_1.
3. Substitute y and t values into the y equation to create a system of equations you can solve for m_2 and b_2.

Examples:

Write parametric equations for the line segment starting at $(1,2)$ with $t = 0$ and ending at $(8,10)$ with $t = 1$.

$x = m_1 t + b_1$ $\quad$ $y = m_2 t + b_2$

$t=0,\ x=1,\ y=2$: $\quad 1 = m_1(0) + b_1$, $\quad 1 = b_1$ $\quad$ $2 = m_2(0) + b_2$, $\quad 2 = b_2$

$t=1,\ x=8,\ y=10$: $\quad 8 = m_1(1) + 1$, $\quad m_1 = 7$ $\quad$ $10 = m_2(1) + 2$, $\quad m_2 = 8$

$$\boxed{\begin{aligned} x &= 7t + 1 \\ y &= 8t + 2 \end{aligned}}$$

Write parametric equations for the line segment starting at $(-2,4)$ with $t = 3$ and ending at $(5,-9)$ with $t = 7$.

$x = m_1 t + b_1$ $\quad$ $y = m_2 t + b_2$

$t=3,\ x=-2,\ y=4$: $\quad -2 = 3m_1 + b_1$ $\quad$ $4 = 3m_2 + b_2$

$t=7,\ x=5,\ y=-9$: $\quad 5 = 7m_1 + b_1$ $\quad$ $-9 = 7m_2 + b_2$

$(-2 = 3m_1 + b_1)(-1)$

$5 = 7m_1 + b_1$

$2 = -3m_1 - b_1$

$7 = 4m_1$

$m_1 = 7/4$

$-2 = 3(7/4) + b_1$

$-2 = 21/4 + b_1$

$b_1 = -29/4$

$(4 = 3m_2 + b_2)(-1)$

$-9 = 7m_2 + b_2$

$-4 = -3m_2 - b_2$

$-13 = 4m_2$

$m_2 = -\frac{13}{4}$

$4 = 3\left(-\frac{13}{4}\right) + b_2$

$4 = -\frac{39}{4} + b_2$

$b_2 = \frac{55}{4}$

$$\boxed{\begin{aligned} x &= \tfrac{7}{4}t - \tfrac{29}{4} \\ y &= -\tfrac{13}{4}t + \tfrac{55}{4} \end{aligned}}$$

Writing Parametric Equations for a Polar Equation

Use the equations $x = r\cos\theta$ and $y = r\sin\theta$. Replace r to obtain the parametric equations. When converting polar equations to parametric equations, θ acts as the parameter.

Example: Write parametric equations for the polar equation $r = 3\cos\theta$.

$x = r\cos\theta$

$x = (3\cos\theta)\cos\theta$

$y = r\sin\theta$

$y = (3\cos\theta)\sin\theta$

$$\boxed{\begin{aligned} x &= 3\cos^2\theta \\ y &= 3\cos\theta\sin\theta \end{aligned}}$$